Perturbation Theory and the Aharonov-Bohm Effect

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Abstract

The perturbation theory expansion of the Aharonov-Bohm scattering amplitude has previously been studied in the context of quantum mechanics for spin zero and spin-1/2 particles as well in Galilean covariant field theory. This problem is reconsidered in the framework of the model in which the flux line is considered to have a finite radius which is shrunk to zero at the end of the calculation. General agreement with earlier results is obtained but with the advantage of a treatment which unifies all the various subcases.

I. Introduction

The Aharonov-Bohm effect [1] is generally considered to be among the more intriguing predictions of quantum mechanics. One particular reason for this is the fact that it requires that the potential itself be viewed as having a physical significance which transcends its role as a mere mathematical construct for the calculation of a classical force. However, there are also some mathematical aspects of this phenomenon which are quite intriguing. Not the least of these is the study of the scattering amplitude in perturbation theory as was pointed out [2] some years ago.

In order to understand the nature of the difficulty it is convenient to refer to the wave equation for the partial wave $f_m(r)$. It can be written as

$$\left[\frac{1}{r} \frac{\partial}{\partial r} r \frac{\partial}{\partial r} + k^2 - \frac{(m + \alpha)^2}{r^2} \right] f_m(r) = 0$$

where α is the flux parameter and $k^2 = 2ME$ with M the particle mass and E its energy. If one applies standard techniques and invokes the condition $f_m(0) = 0$, it is found that the phase shifts can be written as

$$\delta_m = \frac{-\pi}{2} |m + \alpha| + \frac{\pi}{2} |m|$$

so that for small α

$$\delta_0 = -|\alpha|\pi/2 \,.$$

This result is somewhat disturbing since it implies that the $m = 0$ contribution to the scattering amplitude is of order α even though the $m = 0$ potential is proportional to the square of the flux parameter.

This aspect was examined in some detail by Aharonov *et al.* [3] in the framework of a model in which the solenoid was made impenetrable and of finite radius R. Although in the limit $R \to 0$ the exact solution for finite R was found to become identical to the usual Aharonov-Bohm (AB) amplitude, to any finite order in α the solution took the form of a complicated expansion in powers of $\alpha \ell n(kR/2)$. The results of ref. 3 were obtained by

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retaining only that part of the $m = 0$ wave function proportional to $J_\alpha(kr)$ and discarding terms involving the solution $N_\alpha(kr)$ which is singular at the origin. The impropriety of ignoring that part of the solution has been remarked upon by Nagel. In fact the inclusion of such terms affects the quantitative, though not the qualitative, results of ref. 3.

In fact one can anticipate on the basis of fairly general considerations that any perturbative expansion must experience some difficulty. To this end one refers to the (slightly corrected) form of the AB amplitude given in [5]. There one has

$$f_{\text{AB}}(\phi) = - \left(\frac{i}{2\pi k} \right)^{1/2} \frac{\sin \pi \alpha e^{-iN(\phi - \pi)}}{\cos \phi/2} e^{-i\phi/2} \tag{1}$$

where the incident wave is assumed to be from the right and N is the largest integer in α, i.e.,

$$\alpha = N + \beta$$

with

$$0 \le \beta < 1 .$$

For small α one readily obtains from (1) the result

$$f_{\text{AB}}(\phi) \to \alpha \left(\frac{i\pi}{2k} \right)^{1/2} (i \tan \phi/2 - \epsilon(\alpha)) \tag{2}$$

where

$$\epsilon(\alpha) \equiv \alpha/|\alpha| .$$

Evidently, the second term in (2) (which arises from the $m = 0$ wave contribution to the amplitude) is nonanalytic in α, a fact which can be expected to complicate any perturbative expansion of the amplitude in powers of α.

In order to display most effectively the problems associated with a perturbative approach to the Aharonov-Bohm effect, it is crucial to use a unified model which allows one to consider simultaneously both the spinless and spin-1/2 cases. The details of such a model are presented in the following section. Since it is known [6] that the lowest order

Born approximation works well in the spinor case, it is important to establish that the model agrees with that lowest order result and also that it accounts for the difficulties of the spinless case noted in refs. 1-4. In sec. 3 this program is carried out. It is shown there that the perturbative expansion works to all orders in α in the spin-1/2 case for both spin orientations and also that there exist logarithmic singularities in each order for the case that there is no spin degree of freedom. In the Conclusion some general observations are presented and contact made with results which have been obtained in the context of Galilean field theory.

II. The Finite Radius Flux Tube Model

It has already been noted that the perturbative treatment of the AB effect given in ref. 3 was based on an impenetrable solenoid of radius R. Thus the wave function which solves Eq. (1) is a combination of the regular Bessel function $J_{|m+\alpha|}(kr)$ and the irregular one $N_{|m+\alpha|}(kr)$. Specifically

$$f_m(r) \sim \left[N_{|m+\alpha|}(kR)J_{|m+\alpha|}(kr) - J_{|m+\alpha|}(kR)N_{|m+\alpha|}(kr) \right] \ .$$

In the case of AB scattering for spin-1/2 particles, however, it is well known [7] that for a flux tube of zero radius the wave equation (1) must be replaced by

$$\left[\frac{1}{r}\frac{\partial}{\partial r}r\frac{\partial}{\partial r} + k^2 - \frac{(m+\alpha)^2}{r^2} - \alpha s \frac{1}{r}\delta(r) \right] f_m(r) = 0 \tag{3}$$

where $s \pm 1$ is the spin projection parameter. The delta function term evidently describes the interaction of the particle's magnetic moment with the magnetic field of the flux tube. If one were to adopt the impenetrable solenoid model of ref. 3, it is clear that the magnetic moment term would be rendered ineffective and there could be no spin effect whatever. In view of the desire expressed here to develop a unified approach which will accommodate both spin zero and spin-1/2, a rather different model (or regularization) is therefore required.

Such a model has in fact been presented in the context of obtaining the solution of the spin-1/2 AB scattering amplitude [7]. It consists of replacing the zero radius flux tube

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by one of radius R, with the additional condition that the magnetic field be confined to the surface of the tube [8].In this case Eq. (3) is replaced by

$$\left[\frac{1}{r}\frac{\partial}{\partial r}r\frac{\partial}{\partial r} + k^2 - \frac{1}{r^2}[m + \alpha\theta(r - R)]^2 - \alpha s\frac{1}{R}\delta(r - R)\right]f_m(r) = 0 \qquad (4)$$

where $\theta(x)$ is the usual step function

$$\theta(x) = \frac{1}{2}[1 + \epsilon(x)].$$

Since the ultimate interest here is in a perturbation expansion, there is no need to consider any partial wave except $m = 0$, the only case in which a perturbative failure can occur. Thus one writes the solution of (4) as

$$f_0(r) = AJ_\alpha(kr) + BN_\alpha(kr) \qquad (5)$$

for $r > R$ and

$$f_0(r) = CJ_0(kr)$$

for $r < R$. It is to be understood that even though α can be of either sign, it is always to be taken to mean $|\alpha|$ when used to denote the order of a Bessel function. Also worth mentioning is the fact that although the calculations of ref. 7 based on this model used the functions $J_{\pm|m+\alpha|}(kr)$ to describe the $r > R$ solution, these are an inappropriate set for a perturbative analysis. The latter must maintain consistency for $\alpha \to 0$, a requirement clearly violated by the functions $J_{\pm\alpha}(kr)$ which become identical in that limit.

From (4) one obtains the continuity relations

$$f_0(R - \epsilon) = f_0(R + \epsilon)$$

$$R\frac{d}{dR}f_0\Big|_{R-\epsilon}^{R+\epsilon} = \alpha s f_0(R)$$

which are readily seen to imply

$$CJ_0(kR) = AJ_\alpha(kR) + BN_\alpha(kR)$$

$$\left(R\frac{\partial}{\partial R} + \alpha s\right) C J_0(kR) = AR\frac{\partial}{\partial R} J_\alpha(kR) + BR\frac{\partial}{\partial R} N_\alpha(kR).$$

This yields for the ratio of irregular to regular part of the wave function the result

$$B/A = \frac{J_0 R\frac{\partial}{\partial R} J_\alpha - J_\alpha(R\frac{\partial}{\partial R} + \alpha s)J_0}{N_\alpha(R\frac{\partial}{\partial R} + \alpha s)J_0 - J_0 R\frac{\partial}{\partial R} N_\alpha}. \tag{6}$$

Since one is interested in the $R \to 0$ limit, it is permissible to drop all terms which differ from the leading term by one or more powers of kR. Thus (6) becomes

$$B/A = \frac{(R\frac{\partial}{\partial R} - \alpha s)J_\alpha}{(-R\frac{\partial}{\partial R} + \alpha s)N_\alpha}. \tag{7}$$

In order to make contact with perturbation theory one notes that the $m = 0$ contribution to the wave function for $r \to \infty$ is of the form

$$f_0(r) \sim J_0(kr) + f_0 e^{ikr}/r^{1/2}. \tag{8}$$

with the first term in (8) arising from the incident plane wave. Upon comparison of (5) and (8) one finds that

$$A e^{i|\alpha|\pi/2} + B e^{-i|\alpha|\pi/2} = 1.$$

Use is now made of the fact that $f_0(r)$ can be written as

$$f_0(r) = J_0(kr) + \int_0^\infty r'dr' g(r,r')\left[\frac{\alpha^2}{r'^2}\theta(r' - R) + \frac{\alpha s}{R}\delta(r' - R)\right] f_0(r')$$

where the Green's function $g(r, r')$ has the form

$$g(r, r') = -\frac{i\pi}{2} J_0(kr_<) H_0^{(1)}(kr_>)$$

with

$$H_0^{(1)}(x) \equiv J_0(x) + iN_0(x).$$

This allows the $m = 0$ scattering amplitude to be identified as

$$\begin{aligned}
f_0 = -i\left(\frac{\pi}{2ik}\right)^{1/2} \frac{e^{-i\pi|\alpha|/2}}{1 + i(B/A)} &\left\{\alpha s J_0(kR)\left[J_\alpha(kR) + \frac{B}{A}N_\alpha(kR)\right]\right. \\
&\left.+ \alpha^2 \int_R^\infty \frac{dr}{r} J_0(kr)\left[J_\alpha(kr) + \frac{B}{A}N_\alpha(kr)\right]\right\}
\end{aligned} \tag{9}$$

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The result (9) together with (7) allows one in appropriate limits to establish the properties of the scattering amplitude as a function of α and R for both the spin zero and spin-1/2 cases. For the latter one considers both $s = +1$ and $s = -1$ while in the spinless case one merely has to set the spin parameter s equal to zero. The results in these various cases are presented in the following section.

III. The Perturbative Expansion

Since the spin-1/2 case is expected to have the fewest complications in perturbation theory, it is natural to begin with that example. One also anticipates on the basis of ref. 7 that the repulsive (i.e., $\alpha s > 0$) delta function will be particularly simple. Indeed, from (7) one finds that in that situation B/A is zero to the required order in kR and that (9) thus reduces to

$$f_0 = -i \left(\frac{\pi}{2ik} \right)^{1/2} e^{-i|\alpha|\pi/2} \left\{ |\alpha| J_\alpha(kR) + \alpha^2 \int_R^\infty \frac{dr}{r} J_0(kr) J_\alpha(kr) \right\} \tag{10}$$

which by elementary means becomes

$$f_0 = -i \left(\frac{\pi}{2ik} \right)^{1/2} e^{-i|\alpha|\pi/2} \frac{2}{\pi} \sin \frac{|\alpha|\pi}{2}$$

or

$$f_0 = \left(\frac{1}{2\pi ki} \right)^{1/2} \left(e^{-i|\alpha|\pi} - 1 \right) . \tag{11}$$

The most significant aspect of this result from the present perspective is that the dependence upon R has vanished since the leading term in kR goes as a positive power of kR greater than one. Had that power instead been proportional to α an infinite series in $\alpha \ln kR$ would have been encountered in perturbation theory. In fact each of the two terms in (10) contains a term proportional to $(kR)^\alpha$ and it is only as a consequence of a delicate cancellation between the spin term and the non-spin term that the R independent result (11) is obtained.

Somewhat more intricate is the attractive case $\alpha s < 0$. In this circumstance application of (7) yields the result

$$B/A = -\tan \pi |\alpha|. \tag{12}$$

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Insertion of (12) into (10) yields upon evaluation of the integrals

$$f_0 = -i \left(\frac{\pi}{2ik} \right)^{1/2} e^{i|\alpha|\pi/2} \left\{ -2|\alpha| J_\alpha(kR) + |\alpha| \tan \pi|\alpha| N_\alpha(kR) + \frac{2}{\pi} \sin \frac{\pi|\alpha|}{2} \right.$$
$$\left. - \tan \pi|\alpha| \left[\frac{2}{\pi} \cos \frac{\pi\alpha}{2} - R \frac{\partial}{\partial R} N_\alpha(kR) \right] \right\}$$

which is readily reduced to

$$f_0 = \left(\frac{1}{2\pi ki} \right)^{1/2} \left(e^{i\pi|\alpha|} - 1 \right) . \tag{13}$$

Again, a significant cancellation of spin dependent and spin independent terms has resulted in a disappearance of R dependent terms to the order required.

The results (11) and (13) can be combined in the single expression

$$f_0^{s=1/2} = \left(\frac{1}{2\pi ki} \right)^{1/2} \left[e^{-i\alpha\epsilon(s)} - 1 \right]$$

It confirms the lowest order Born approximation result of ref. 6 by virtue of its independence of the flux tube radius as well as by its analyticity in the flux parameter. Clearly, perturbation theory works to all orders for spin-1/2.

These results are significantly changed in the $s = 0$ case. Application of (7) and (9) yields.

$$f_0^{s=0} = -i \left(\frac{2}{\pi ik} \right)^{1/2} e^{-i|\alpha|\pi/2} \frac{1 + \frac{B}{A} \cot \frac{|\alpha|\pi}{2}}{1 + \frac{iB}{A}} \sin \frac{\pi|\alpha|}{2} \tag{14}$$

where

$$B/A = -\frac{R \frac{\partial}{\partial R} J_\alpha(kR)}{N \frac{\partial}{\partial R} N_\alpha(kR)} . \tag{15}$$

Upon taking the limit $R \to 0$ for fixed α one readily finds that $B/A \to 0$ and that

$$f_0^{s=0} \to \left(\frac{1}{2\pi ki} \right)^{1/2} \left(e^{-i|\alpha|\pi} - 1 \right) \tag{16}$$

which is precisely the usual $m = 0$ contribution to the spinless AB scattering amplitude. As expected, it coincides with (11), the result in the spin-1/2 case for a repulsive magnetic moment interaction.

On the other hand an expansion of Eqs. (14) and (15) in powers of α is plagued with logarithmic singularities. In particular the lowest nonvanishing contribution gives

$$f_0^{s=0} \cong \left(\frac{i\pi}{2k}\right)^{1/2} \alpha^2 (\ell n \frac{kR}{2} + \gamma)$$

where γ is Euler's constant. It is a curious fact that that this happens to agree to lowest order with the result of ref. 3 despite the fact that the model considered there is physically quite different. Also relevant in this context is Nagel's observation that the lowest order result of ref. 3 is too large by a factor of 3.

IV. Conclusion

In this work the perturbative expansion of the AB scattering amplitude has been considered from the viewpoint of the finite radius flux tube model. The latter has successfully been applied in the past to the problem of calculating exactly the AB amplitude for both spin zero and spin-1/2 scattering. Thus it is not surprising that it is also able to deal with the perturbative approach to this problem. In the spinor case the lowest order result of ref. 7 has been confirmed and extended to arbitrary order in α. For spinless particles it has been shown that the AB amplitude is an infinite series in $\alpha \ell n kR$ which can be summed to give the known form of that amplitude.

It is also appropriate to discuss briefly the relevance of this development to the corresponding problem in Galilean field theory. The framework for such an approach was provided in ref. 9 which formulated the Galilean covariant gauge theory of the Chern-Simons interaction. Because of the superselection rule on the mass, it was shown there that one could approach the problem of calculating an arbitrary scattering process by restricting consideration to an N-body sector. This allowed one to derive a Schrodinger equation for the N body problem with each pair interacting as zero radius flux tubes. Thus the field theory sector by sector is formally equivalent to obtaining a solution of a conventional Schrodinger equation. However, once that set of Schrodinger equations is obtained, it is necessary to give it a precise meaning by either stipulating a set of boundary conditions or by regularizing the interaction. In the latter case one can invoke the

impenetrable solenoid of ref. 3 or the finite radius flux tube model of ref. 7. The latter, of course, is more flexible since it allows for the accommodation of spin. In view of the equivalence between the field theory and the Schrodinger equation approaches, it is clear that all the calculations of this paper apply to both domains. What would be interesting and worthwhile in this context would be to write the field theory of ref. 9 *ab initio* in such a way that the methodology of the finite radius flux tube method is built in at the start. This can presumably be done by a point splitting definition of the current operator, but has not yet been carried out.

Another way in which one can approach the field theoretical approach to the AB scattering calculation is to determine the propagators of the theory and subsequently use them to carry out perturbative calculations. This has recently been done by Bergman *et al.* [10] who find that for spinless particles the AB amplitude is logarithmically divergent. On the basis of renormalizability considerations they argue for the inclusion of a contact interaction which allows the logarithmic divergence to be eliminated. Since the latter type of interaction is formally identical to the $\alpha s 1/r \delta(r)$ term of the Schrodinger equation for spin-$1/2$, it is clear that the cancellation found in ref. 10 to lowest order coincides with the repulsive (i.e., $\alpha s > 0$) case of the preceding section. It is interesting to note, however, that the case $\alpha s < 0$ also leads to such a cancellation. This has been shown here in general and can also be verified using the lowest order calculations of ref. 10. In that work the choice of a repulsive contact term was made in order to obtain agreement in perturbation theory with the AB amplitude. The opposite choice is, however, equally allowable and gives not the AB amplitude but rather an AB-like amplitude in which $|\alpha|$ is simply replaced by its negative in Eq. (16).

A final comment (which applies to the sector-by-sector approach as well) has to do with the details of the cancellation of divergences between the spin (or spinlike) term and the α^2/r^2 potential. These were shown to cancel unambiguously in the finite radius flux tube model. However, in the calculations carried out in ref. 10 it is reasonable – but not

compelling – to identify the cutoffs associated with the two parts of the interaction. If their ratio is different than unity, one must have additional finite terms in the amplitude which destroy the exact agreement with the perturbative AB amplitude. Thus in the absence of a Galilean field theory which is well-defined (i.e., unambiguously regularized) at the outset, the perturbative approach to AB scattering must remain only inadequately understood in that case relative to the corresponding quantum mechanical result.

Acknowledgements

References

1. Y. Aharonov and D. Bohm, Phys. Rev. **115**, 485 (1959).

2. E. L. Feinberg, Usp. Fiz. Nauk. **78**, 53 (1962) [Sov. Phys. Usp. **5**, 753 (1963)]; E. Corinaldesi and F. Rafeli, Ann. J. Phys. **46**, 1185 (1978).

3. Y. Aharonov, C. K. Au, E. C. Lerner, and J. Q. Liang, Phys. Rev. **D29**, 2396 (1984).

4. B. Nagel, Phys. Rev. **D32**, 3328 (1985).

5. C. R. Hagen, Phys. Rev. **D41**, 2015 (1990).

6. F. Vera and I. Schmidt, Phys. Rev. **D42**, 3591 (1990).

7. C. R. Hagen, Phys. Rev. Lett. **64**, 503 (1990).

8. It has been shown in ref. 7 that in the $R = 0$ limit the result is independent of the precise distribution of the magnetic field inside the tube. However, a surface distribution of the field is a convenient assumption since it considerably simplifies the calculations involved.

9. C. R. Hagen, Phys. Rev. **D31**, 848 (1985).

10. O. Bergman and G. Lozano, Ann. Phys. **229**, 416 (1994); D. Bak, O. Bergman, Phys. Rev. **D15**, 1994 (1995).